

Antifactor of regular bipartite graphs*

Hongliang Lu[†]

^aDepartment of Mathematics

Xi'an Jiaotong University, Xi'an 710049, PR China

Abstract

Let G be a graph and let $H : V(G) \rightarrow 2^N$ be a set function. An H -factor of G is a spanning subgraph F such that $d_F(v) \in H(v)$ for all $v \in V(G)$. The H -factor problem is to find an H -factor of graph G . This problem is NP-complete in general, but for the case when no prescription contains two consecutive gaps, Lovász gave a structural description, and Cornuéjols gave a polynomial algorithm. However, results on H -factors are known only in some special cases, such as parity intervals or general antifactors. Let $k \geq 3$ be an integer. Let G' be a k -regular bipartite graph with partition (X, Y) . In this paper, we show that G' contains an H -factor, where $H(x) = \{1\}$ for all $x \in X$ and $H(y) = \{0, 2, 3, \dots, k\}$ for all $y \in Y$, which solves a problem proposed by Liu and Yu.

1 Introduction

In this paper, we consider finite undirected graphs without loops and multiple edges. For a graph $G = (V, E)$, the degree of x in G is denoted by $d_G(x)$, and the set of vertices adjacent to x in G is denoted by $N_G(x)$. For $x \in V(G)$, we write $N[x] = N(x) \cup \{x\}$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$ and $G - S = G[V(G) - S]$. For vertex subsets S and T , $E_G(S, T)$ is the set of edges between S and T in G . Let $e_G(S, T) = |E_G(S, T)|$.

Let G be an arbitrary graph, and let a set $H(x)$ of non-negative integers be associated with every vertex $x \in V(G)$. An H -factor is a spanning graph F

$$d_F(x) \in H(x) \quad \text{for all } x \in V(G). \quad (1)$$

By specifying $H(x)$ to be an interval or a special set, an H -factor becomes an k -factor, an f -factor, an $[a, b]$ -factor or a (g, f) -factor, respectively. In particular, if $k = 1$, then

*This work is supported by the National Natural Science Foundation of China (Grant No. 11471257)

[†]Corresponding email: luhongliang215@sina.com (H. Lu)

a k -factor is called 1 -factor. For 1 -factor in bipartite graphs, Hall obtained the following result.

Theorem 1.1 (Hall, [3]) *Let G be a regular bipartite graph. Then G contains a 1 -factor.*

Lovász [4] obtained a sufficient and necessary condition of “general factors”, that is, H -factors, where H satisfies the following property for all $x \in V(G)$:

$$(*) \quad \text{if } i \notin H(x), \text{ then } i + 1 \in H(x), \text{ for } mH(x) \leq i \leq MH(x),$$

where $mH(x) = \min\{r \mid r \in H(x)\}$ and $MH(x) = \max\{r \mid r \in H(x)\}$. Let $MH(S) = \sum_{u \in S} MH(u)$, $mH(S) = \sum_{v \in S} mH(v)$ and $H \pm c = \{i \pm c \mid i \in H\}$.

A set function H associated with G is called an *allowed set function* (see [4]) if $H(v)$ satisfies Property $(*)$ all $v \in V(G)$. Lovász [4] showed that if H is not an allowed set function, then the decision problem of determining whether a graph has an H -factor is known to be NP -complete.

Let G be an bipartite graph with bipartition (X, Y) . Let L be a set function such that $L(x) = \{1\}$ for all $x \in X$ and $L(y) = N - \{1\}$. An L -factor of bipartite graph is called *1-antifactor*. 1 -antifactor is an instance of “general factors”. Let G_0 be any graph, $A = V(G_0)$ and let B be the set of all edges and triangles of G_0 . Let G be the bipartite graph with bipartition (A, B) and all edges of the form xy with $x \in V(y)$. Then G has a 1 -antifactor if and only if G_0 contains a set of vertex-disjoint edges and triangles covering $V(G_0)$. This latter problem was solved by Cornuéjols, Hartvigsen and Pulleyblank [1].

Lovász and Plummer (see [5], Page 390, Line 2) asked the following problem.

Problem 1.2 *Let G be a bipartite graph with bipartition (X, Y) . Can one find a polynomial algorithm for the existence of H -factor, where $H(x) = 1$ for any $x \in A$ and $H(y) = N - \{1\}$?*

Cornuéjols [2] provided the first polynomial algorithm for the problem with H being allowed and so give a affirmative answer for Problem 1.2. However, it is not easy to determine whether a bipartite graph contains 1 -antifactor. So it is interesting to classify the bipartite graph with 1 -antifactor. Liu and Yu (see [6], Page 76, Line 1) proposed the following problem.

Problem 1.3 *Let $k \geq 3$ and let G be a k -regular bipartite graph with bipartition (X, Y) . Does G contains an H -factor, where $H(x) = \{1\}$ for all $x \in X$ and $H(y) = N - \{1\}$ for all $y \in Y$?*

In this paper, we give an affirmative answer for Problem 1.3 and obtain the following result.

Theorem 1.4 *Let ≥ 3 be an integer and let G be a k -regular bipartite graph with bipartition (X, Y) . G contains an H -factor, where $H(x) = \{1\}$ for all $x \in X$ and $H(y) = N - \{1\}$ for all $y \in Y$.*

For the proof of Theorem 1.4, we need introduce Lovasz's H -Factor Structure Theorem.

2 Lovasz's H -Factor Structure Theorem

Let F be a spanning subgraph of G and let $H : V(G) \rightarrow 2^N$ be an allowed set function. Following Lovász [4], one may measure the “deviation” of F from the condition (1) by

$$\nabla_H(F; G) = \sum_{v \in V(G)} \min\{|d_F(v) - h| : h \in H(v)\}. \quad (2)$$

Moreover, the “solvability” of (1) can be characterized by

$$\nabla_H(G) = \min\{\nabla_H(F) : F \text{ is a spanning subgraph of } G\}.$$

The subgraph F is said to be H -optimal if $\nabla_H(F; G) = \nabla_H(G)$. It is clear that F is an H -factor if and only if $\nabla_H(F; G) = 0$, and any H -factor (if exists) is H -optimal. Let

$$Q = \{h_1, h_2, \dots, h_m\},$$

where $h_1 < h_2 < \dots < h_m$.

We study H -factors of graphs based on Lovász's structural description to the degree prescribed subgraph problem. For $v \in V(G)$, we denote by $I_H(v)$ the set of vertex degrees of v in all H -optimal subgraphs of graph G , i.e.,

$$I_H(v) = \{d_F(v) : \text{all } H\text{-optimal subgraphs } F\}.$$

Based on the relation of the sets $I_H(v)$ and $H(v)$, one may partition the vertex set $V(G)$ into four classes:

$$C_H = \{v \in V(G) : I_H(v) \subseteq H(v)\},$$

$$A_H = \{v \in V(G) - C_H : \min I_H(v) \geq \max H(v)\},$$

$$B_H = \{v \in V(G) - C_H : \max I_H(v) \leq \min H(v)\},$$

$$D_H = V(G) - A_H - B_H - C_H.$$

It is clear that the 4-tuple (A_H, B_H, C_H, D_H) is a partition of $V(G)$. We call it the H -decomposition of G . In fact, the four subsets can be distinguished according to the contributions of their members to the deviation (2). A graph G is said to be H -critical if it is

connected and $D_H = V(G)$. For non-consecutive allowed set function, the only necessary condition of H -critical graph is given by Lovász [4]. In this paper, we obtain a sufficient condition for a graph to be not H -critical.

We write $MH(x) = \max H(x)$ and $mH(x) = \min H(x)$ for $x \in V(G)$. For $S \subseteq V(G)$, let $MH(S) = \sum_{x \in S} MH(x)$ and $mH(S) = \sum_{x \in S} mH(x)$. By the definition of A_H, B_H, C_H, D_H , the following observations hold:

- (I) for every $x \in B_H$, there exists an H -optimal graph F such that $d_F(x) < mH(x)$;
- (II) for every $x \in A_H$, there exists an H -optimal graph F such that $d_F(x) > MH(x)$;
- (III) for every $x \in D_H$, there exists an H -optimal graph F such that $d_F(x) < MH(x)$ and other H -optimal graph F' such that $d_{F'}(x) > mH(x)$.

Lovász [4] gave the following properties.

Lemma 2.1 (Lovász, [4]) $E_G(C_H, D_H) = \emptyset$.

Lemma 2.2 (Lovász, [4]) *If G is a simple graph, then $I_H(v)$ is an interval for all $v \in D_H$.*

Lemma 2.3 (Lovász, [4]) *The intersection $I_H(v) \cap H(v)$ contains no consecutive integers for any vertex $v \in D_H$.*

Given an integer set P and an integer a , we write $P - a = \{p - a \mid p \in P\}$. Let R be a connected induced subgraph of G and $T \subseteq V(G) - V(R)$. Let $H_{R,T} : V(R) \rightarrow 2^N$ be a set function such that $H_{R,T}(x) = H(x) - e_G(x, T)$ for all $x \in V(R)$.

Lemma 2.4 (Lovász, [4]) *Every component R of $G[D_H]$ is H_{R,B_H} -critical and if F is H -optimal, then $F[V(R)]$ is H_{R,B_H} -optimal.*

Lemma 2.5 (Lovász, [4]) *If G is H -critical, then $\nabla_H = 1$.*

Theorem 2.6 (Lovász, [4]) $\nabla_H(G) = \omega(G[D_H]) + \sum_{v \in B_H} (mH(v) - d_{G-A_H}(v)) - \sum_{v \in A_H} MH(v)$.

3 The Proof of Theorem 1.4

Let G be a bipartite graph with bipartition (X, Y) and let $H, L : V(G) \rightarrow 2^N$ such that $H(y) = L(y)$ for all $y \in Y$, $H(x) = 1$ and $L(x) = \{-1, 1\}$. In next discussion of this paper, we fix the notations G , L and H . It is easy to see that G contains an L -factor if and only if G also contains an H -factor.

Lemma 3.1 *If G is L -critical, then $|X|$ is odd and for any $x \in X$, $G - x$ contains an L -factor.*

Proof. Suppose that G is H -critical. By the definition of L -critical graph and Lemma 2.4, one can see that $\nabla_L(G) = 1$ and $D_L = V(G)$. For any $v \in X$, by the definition of D_L , there exists an L -optimal subgraph F of G such that $d_F(x) = 0$ and $d_F(w) \in H(w)$ for all $w \in V(G) - x$. Hence $G - x$ contains an L -factor. By Lemmas 2.3 and 2.2, $I_L(v)$ is an internal and $I_L(v) \cap L(v)$ contains no two consecutive integers. Combining the definition of D_L , one can see that $I_L(x) - L(x) \neq \emptyset$ and $I_L(x) \subset \{0, 1, 2\}$ for all $x \in V(G)$. Thus one can see that $d_F(u) = 1$ for all $u \in X - x$ and $d_F(y) \in \{0, 2\}$ for all $y \in Y$. By parity, we infer that $|X|$ is odd. This completes the proof. \square

Theorem 3.2 *G contains an L -factor if and only if for any subset $S \subseteq X$ such that*

$$q(G - S) \leq |S|,$$

where $q(G - S)$ denotes the number of components R of $G - S$, called L -odd components, such that R is L -critical.

Proof. Firstly, we prove the necessity. For any $S \subset X$, let $q(G - S)$ denote the number of L -critical components of $G - S$ and let C_1, \dots, C_q denote these L -critical components of $G - S$. Since C_i contains no L -factors, every L -factor F of H contains at least an edge from C_i to S . Thus

$$q(G - S) \leq \sum_{x \in S} d_F(x) = |S|,$$

which implies $q(G - S) \leq |S|$.

Now we prove the sufficiency. Conversely, suppose that G contains no L -factor. Let A_L, B_L, C_L, D_L be defined as Section 2.

Claim. 1. $A_L \cup B_L \subseteq X$.

By the definition of set A_L , we have $\min\{I_L(x)\} \geq ML(x)$ for all $x \in A_L$. If $x \in A_L \cap Y$, then we have $\min\{I_L(x)\} \geq ML(x) = d_G(x)$, which implies $I_L(x) \subseteq L(x)$, a contradiction.

Claim. 2. $B_L = \emptyset$.

Conversely, suppose that $B_L \neq \emptyset$ and let $v \in B$. By the definition of B_L , if $v \in X$, then $\max I_L(v) \leq \min L(v) = -1$, which is impossible. Thus we may assume that $v \in Y$. This implies that $\max I_L(v) \leq \min L(v) = 0$. Hence $I_L(v) \subset L(v)$, which implies $v \in C$, a contradiction.

By Theorem 2.6, one can see that

$$\begin{aligned} 0 < \nabla_L(G) &= \omega(G[D_L]) + \sum_{v \in B_L} (mL(v) - d_{G-A_L}(v)) - \sum_{v \in A_L} ML(v) \\ &= \omega(G[D_L]) - |A_L|, \end{aligned}$$

where $\omega(G[D_L])$ denotes the number of components R of $G - A_L$ without $L_{R,\emptyset}$ -factors, i.e., the number of components R of $G - A_L$ without L -factors. Now we get

$$\omega(G[D_L]) < |A_L|,$$

a contradiction. This completes the proof. \square

Lemma 3.3 *Let $r \geq 3$ be an integer and let G be a k -regular bipartite graph with bipartition (X, Y) . Either G contains a L -factor or G is L -critical.*

Proof. Suppose that G contains no L -factor and is not L -critical. Let A_L, B_L, C_L and D_L be defined as Section 2. By Theorem 2.6,

$$q_L(A_L, B_L) - \sum_{x \in B_L} d_{G-A_L}(x) - ML(A_L) + mL(B_L) > 0, \quad (3)$$

where q_L denotes the number of components R of $G - A_L - B_L$ such that R is L_{R,B_L} -critical. Since G is not L -critical, $A_L \cup B_L \neq \emptyset$. By the definition of A_L (B_L), for every $v \in A_L$ ($v \in B_L$), there exists a H -optimal subgraph F such that $d_F(v) > ML(v)$ ($d_F(v) < mL(v)$, respectively). With the same proof as Theorem 3.2 Claims 1 and 2, we may infer that

$$B_L = \text{ and } A_L \subset X. \quad (4)$$

Let C_1, \dots, C_{q_L} denote the components of $G - A_L - B_L$ such that C_i contains no $H_{(A_L, B_L)|C_i}$ -factors for $i = 1, \dots, q_L$. Since G is a k -regular connected bipartite graph and by (4), $d_{C_i}(x) = k$ for all $x \in V(C_i) \cap X$. So we infer that

$$|C_i \cap X| < |C_i \cap Y|.$$

So we have $e(V(C_i), A_L \cup B_L) \geq k$. Since G is k -regular, $kq_L(A_L, B_L) \leq k|A_L|$, and

$$q_L(A_L, B_L) \leq |A_L|. \quad (5)$$

By (3), we have

$$\begin{aligned} \nabla_L(G) &= q_L(A_L, B_L) - \sum_{x \in B_L} d_{G-A_L}(x) - MH(A_L) + mH(B_L) \\ &\geq q_L(A_L, B_L) - |A_L| \\ &= q_L(A_L, B_L) - |A_L| > 0. \end{aligned}$$

which implies $|A_L| < q_L(B_L, A_L)$, contradicting (5). This completes the proof. \square

Let \mathcal{H} be the set of graphs G , which satisfying the following property:

- (a) G is a connected bipartite graph with bipartition (X, Y) ;
- (b) $d_G(u) = 3$ for every vertex $x \in X$ and $d_R(y) \leq 3$ for every vertex $y \in Y$.

Lemma 3.4 *If $G \in \mathcal{H}$, then G is not L -critical.*

Proof. Suppose that the result does not hold. Let $G \in \mathcal{H}$ be L -critical graph with the smallest order. By Lemma 3.1, we may assume that $|X|$ is odd. If $|X| = 3$, then $|Y| = 4$, there exists a vertex $w \in Y$ such that $d_G(w) = 3$. Hence G contains an L -factor, a contradiction.

Hence we can assume that $|X| \geq 5$. By the definition of G , either G contains three vertices of degree two or it contains one vertex of degree one and one vertex of degree two. Now we discuss two cases.

Case. 1. G contains three vertices of degree two.

Since $|Y| \geq |X| + 1 \geq 6$, then there exists a vertex $y \in Y$ such that $d_G(y) = 3$. We write $N(y) = \{x_1, x_2, x_3\}$. Since G is L -critical, by Lemma 2.3, $I_L(y) \cap L(y)$ contains no two consecutive integers and there exists an L -optimal subgraph F of G such that $d_F(y) \notin L(y)$. By Theorem 2.2, $I_L(y)$ is an interval, which implies $I_L(y) \subseteq \{0, 1, 2\}$, which means that $3 \notin I_L(y)$. We claim that the deficiency of $G - N[y]$ is equal to two. Otherwise, there exists an L -optimal subgraph F' of $G - N[y]$ such that $\nabla_L(F'; G - N[y]) = 1$. Thus $F' \cup G[N[y]]$ is an L -optimal subgraph of G , contradicting to $d_{F' \cup G[N[y]]} = 3 \notin I_L(y)$.

By Theorem 3.2, there exists a subset $S \subseteq U$ such that

$$q(G - N[y] - S) = |S| + 2,$$

where $q(G - N[x] - S)$ denote the number of L -critical components of $G - S$.

Now we show that $G - N[x] - S$ contains a component T such that $T \in \mathcal{H}$ and T is L -critical, which results a contradiction. We write $|S| = s$ and let C_1, \dots, C_{s+2} denote these L -critical components of $G - N[x] - S$ and let $C_{s+3}, \dots, C_{s+2+r}$ denote the components of $G - N[y] - S - \cup_{i=1}^{s+2} V(C_i)$. By Lemma 3.1, $|V(C_i) \cap X|$ is odd for $i = 1, \dots, s + 2$.

Since G is connected, then $e_G(V(C_i), S \cup N(y)) > 0$, for $i = 1, \dots, s + 2 + r$. Note that $S \subseteq X$, so $d_{C_i}(v) = d_F(v)$ for every $v \in V(C_i) \cap X$. Hence we have $|V(C_i) \cap X| \leq$

$|V(C_i) \cap Y| = 1$ for $i = 1, \dots, s + r + 2$, which implies

$$\begin{aligned} |X| &= s + \sum_{i=1}^{s+2+r} |V(C_i) \cap X| + |N(y)| \\ &= s + \sum_{i=1}^{s+2+r} |V(C_i) \cap X| + 3 \\ &\leq \sum_{i=1}^{s+2+r} |V(C_i) \cap Y| + 1 - r, \end{aligned}$$

i.e.,

$$|X| \leq \sum_{i=1}^{s+2+r} |V(C_i) \cap Y| + 1 - r. \quad (6)$$

Note that

$$|X| + 1 = Y = \sum_{i=1}^{s+2+r} |V(C_i) \cap Y| + 1 \quad (7)$$

Combining (6) and (7), we infer that $r \leq 1$. Let u_1, u_2, u_3 be three vertices of degree two of G . Now we discuss two subcases.

Subcase. 1.1. $r = 1$.

By (6) and (7), one can see that $|V(C_i) \cap X| = |V(C_i) \cap Y| - 1$ for $i = 1, \dots, s + 3$. Since $E_G(V(C_i) \cap X, S \cup N(y)) = \emptyset$, then we have

$$e(C_i) = \sum_{x \in V(C_i) \cap X} d_F(x) = 3|V(C_i) \cap X|.$$

Hence we have $C_i \in \mathcal{H}$, which contradicts the choice of G since $|V(C_2)| < |V(G)|$.

Subcase. 1.2. $r = 0$.

Since $|X| = |Y| - 1$, then $G - N[x]$ contains one component, say C_1 such that $|V(C_i) \cap X| \leq |V(C_i) \cap Y| - 2$ and $|V(C_i) \cap X| = |V(C_i) \cap Y| - 1$ for $i = 2, \dots, s + 2$. Hence we have $C_2 \in \mathcal{H}$ and C_2 is L -critical, which contradicts to the choice of G .

Case. 2. G contains one vertex of degree one and one vertex of degree two.

Let $y' \in Y$ be a vertex of degree one and $y'x \in E(G)$. There exists a vertex of degree three in $N(x)$, say y . Since G is L -critical, then the deficiency of $F - N[y]$ is equal to two. By Theorem 3.3, there exists a subset $S \subseteq V(G) - N[y]$ such that

$$q(G - S - N[y]) = |S| + 2,$$

where $q(G - S - N[y])$ denote the number of L -critical components of $G - S - N[y]$. Clearly, y' is an isolated vertex of $G - S - N[y]$. Let C_1, \dots, C_{s+2} denote these L -critical components of $G - N[y] - S$.

Suppose that $G - \bigcup_{i=1}^{s+2} V(C_i) - S - (N[y] \cup \{y'\})$ contains r components, say $C_{s+3}, \dots, C_{s+r+2}$. Since G is connected and $S \subseteq X$, then $d_{C_i}(y) = d_G(y) = 3$ for all $y \in V(C_i) \cap X$. Hence we have $|V(C_i) \cap X| \leq |V(C_i) \cap Y| - 1$, which implies

$$\sum_{i=1}^{s+r+2} |V(C_i) \cap X| \leq \sum_{i=1}^{s+r+2} |V(C_i) \cap Y| - (s + r + 2). \quad (8)$$

Note that

$$|X| = \sum_{i=1}^{s+r+2} |V(C_i) \cap X| + |N(y)| + s = \sum_{i=1}^{s+r+2} |V(C_i) \cap X| + s + 3,$$

and

$$|Y| = \sum_{i=1}^{s+r+2} |V(C_i) \cap Y| + \{y, y'\} = \sum_{i=1}^{s+r+2} |V(C_i) \cap Y| + 2.$$

Since $|X| = |Y| - 1$, we have

$$\sum_{i=1}^{s+r+2} |V(C_i) \cap X| + 2 + s = \sum_{i=1}^{s+r+2} |V(C_i) \cap Y|. \quad (9)$$

Combining (8) and (9), we infer that $r = 0$ and $|V(C_i) \cap X| = |V(C_i) \cap Y| - 1$ for $i = 1, \dots, s + 2$. Hence we have $C_1 \in \mathcal{H}$, which contradicts to the choice of G since C_1 is L -critical. This completes the proof. \square

Proof of Theorem 1.4. Let $k \geq 3$ and let G be a connected k -regular bipartite graph. By Hall's Theorem, G contains a 3-factor. Thus it is sufficient for us to show that every 3-regular graph contains an L -factor. So we may assume that $k = 3$. Suppose that G contains no L -factor.

By Lemma 3.3, either G contains an L -factor or G is L -critical. So we infer that G is L -critical. By Lemma 3.1, $|X|$ is odd. Let $y \in Y$ and $G' = G - N[y]$.

Claim. 1. $3 \notin I_L(y)$.

Otherwise, since $I_L(y)$ is an internal, then either $I_L(y) = \{3\}$ or $\{2, 3\} \subset I_L(y)$. By the definition of D_L , one can see that $I_L(y) - H(y) \neq \emptyset$. Thus we infer that $\{2, 3\} \subset I_L(y) \cap L(y)$, contradicting Lemma 2.3.

Since G is L -critical, there exists an L -optimal subgraph F such that $d_F(v) = 2$. Thus we infer that $\nabla_L(G') \leq 2$. By Claim 1, one can see that $\nabla_L(G') = 2$.

By Theorem 3.2, there exists $S \subset X - N(y)$ such that

$$q(G' - S) \geq |S| + 2,$$

where $q(G' - S)$ denote the number of components R such that R is L -critical. We write $q = q(G' - S)$ and let R_1, \dots, R_q denote these L -critical components of $G' - S$. For every $x \in V(R_i) \cap X$, one can see that $d_G(x) = d_{R_i}(x) = 3$. Since G is 3-regular, one can see that $|V(R_i) \cap X| < |V(R_i) \cap Y|$. Thus we infer that

$$e_G(S \cup N(y), V(R_i) \cap Y) = e_G(S \cup N(y), V(R_i)) \geq 3.$$

Therefore, one can see that

$$3(|S| + 2) \leq \sum_{i=1}^q e_G(S \cup N(y), V(R_i) \cap Y) \leq \sum_{x \in S \cup N(y)} d_{G-y}(x) = 3|S| + 6.$$

This equality implies $e_G(S \cup N(y), V(R_i)) = 3$ and $|V(R_i) \cap X| = |V(R_i) \cap Y| - 1$ for $1 \leq i \leq q$. Since R_i is L -critical, R_i is not an isolated vertex. Thus we have $R_i \in \mathcal{H}$, contradicting Lemma 3.4.

This completes the proof. \square

Remark 1. The bound of Theorem 1.4 is sharp. Let $m \in \mathbb{N}$ be an positive integer. For example, C_{4m+2} is a 2-regular graph and contains no an H -factor. However, it is easy to show that C_{4m} contains such an H -factor.

Remark 2. There exist families of infinite bipartite graphs (e.g., the generalized θ -graph) which contains no L -factor. In light of Theorem 1.4, it remains an open problem to classify all bipartite graphs which contain an L -factor.

References

- [1] G. Cornu  jols, D. Hartvigsen and W.R. Pulleyblank, Packing subgraphs in a graph, *O. R. Letters*, **1** (1982), 139–143.
- [2] G. Cornu  jols, General factors of graphs, *J. Combin. Theory Ser. B*, **45** (1988), 185–198.
- [3] P. Hall, On representatives of subsets, *J. London Math. Soc.*, **10** (1935), 26–30.
- [4] L. Lov  sz, The factorization of graphs. II, *Acta Math. Hungar.*, **23** (1972), 223–246.
- [5] L. Lov  sz and M. D. Plummer, Matching Theory, *Ann. Discrete Math.*, **29** North-Holland, Amsterdam, 1986.

- [6] Q. Yu and G. Liu, *Graph Factors and Matching Extensions*, Springer, Berlin, 2009.
- [7] A. Sebö, General antifactors of graphs, *J. Combin. Theory Ser. B*, **58** (1993), 174-184.
- [8] J. Szabó Good characterizations for some degree constrained subgraphs, *J. Combin. Theory Ser. B*, **99** (2009), 436C446.